IIT-JEE 2003 Mains Questions & Solutions - Maths
(The questions are based on memory)

Break-up of questions:

Algebra	Trigonometry	Co-ordinate Geometry	Calculus	Vector/3D
8	1	1	8	2

1. Prove that $\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1$ if $|z_1| < 1 < |z_2|$

[2]

Sol. T.P.T. $\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1$

or, T.P.T. $|1-z_1\bar{z}_2|<|z_2-z_1|$

or, T.P.T. $(1-z_1\bar{z}_2)(1-\bar{z}_1z_2)<(z_2-z_1)(\bar{z}_2-\bar{z}_1)$

or, T.P.T. $1+|z_1|^2|z_2|^2-|z_1|^2-|z_2|^2<0$

or, T.P.T. $(1-|z_1|^2)+(|z_1|^2-1)|z_2|^2<0$

or, T.P.T. $(1-|z_1|^2)(1-|z_2|^2)<0$

Which is true because of $|z_1| < 1 < |z_2|$

2. P(x) is a polynomial function such that P(1) = 0, P′(x) > P(x), ∀ x > 1. Prove that P(x) > 0, ∀ x > 1

[2]

Sol. $P'(x) - P(x) > 0, \forall x > 1$

⇒ $e^{-x}.P'(x)-e^{-x}P(x)>0$, ∀ x > 1 (multiplying by e^{-x} which is +ve)

⇒ $\frac{d}{dx}(e^{-x}.p(x))>0, \forall x>1$

⇒ $e^{-x}.P(x)$ is an increasing function of x, ∀ x ∈ [1, ∞) (as P(x) being polynomial function is a continuous function).

Thus for x > 1

⇒ $e^{-x}.P(x) > e^{-1}.P(1)$

⇒ $e^{-x}.P(x) > 0$, as P(1) = 0

⇒ P(x) > 0. (as e^{-x} +ve)

3. If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, abc =1, $A^T A = I$, then find the value of $a^3 + b^3 + c^3$.

[2]

Sol. $A^T A = I$

$\Rightarrow$ (det A) (det A^T) = 1 $\Rightarrow$ (det A)^2 = 1 $\Rightarrow$ det A = ± 1.

Now det A = –($a^3 + b^3 + c^3$ – 3abc) = –($a^3 + b^3 + c^3$) + 3

Thus $-(a^3+b^3+c^3)+3 = \pm 1 = k(say)$ (say)

$\Rightarrow$ $a^3 + b^3 + c^3$ = 3 – k = 2 or 4.

4. Find the point on $x^2 + 2y^2 = 6$, which is nearest to the line x + y = 7.

[2]

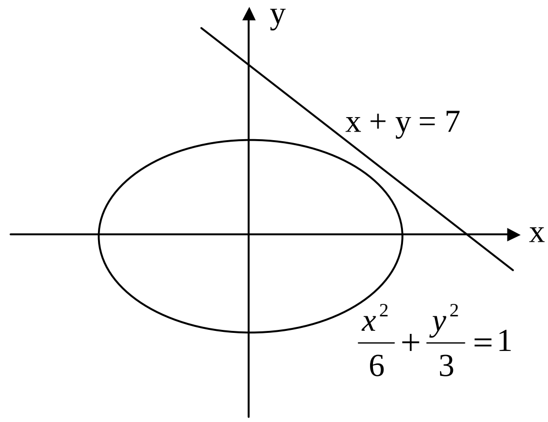
Sol. Let P ≡ ($\sqrt{6}$ cosθ, $\sqrt{3}$ sinθ) be the required point

Tangent at P should be of slope = –1 (slope of the line x + y = 7)

Now $x^2 + 2y^2 = 6$

$\Rightarrow$ $x + 2y\frac{dy}{dx} = 0$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{x}{2y}$



$\Rightarrow$ $\left[\frac{dy}{dx}\right]_P = -\frac{\sqrt{6}\cos\theta}{2\sqrt{3}\sin\theta} = -\frac{1}{\sqrt{2}}\cot\theta = -1$

$\Rightarrow$ cotθ = $\sqrt{2}$

$\Rightarrow$ sinθ = $\frac{1}{\sqrt{3}}$, cosθ = $\sqrt{\frac{2}{3}}$ (as P lies in the I'st quadrant)

Then $P \equiv \left(\sqrt{6}.\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{3}.\frac{1}{\sqrt{3}}\right) \equiv (2, 1)$

5. Prove that $2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1}.......(-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$ where $\binom{n}{m} = {}^nC_m$.

[2]

Sol. $2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1}...(-1)^k\binom{n}{k}\binom{n-k}{0}$

= coefficient of x^k in $\left[{}^nC_0(1+2x)^n - {}^nC_1 x(1+2x)^{n-1} + ...\right]$

= coefficient of x^k in $[(1+2x)-(x)]^n$
= coefficient of x^k in $(1 + x)^n$
= ^nC_k.

6. If $|a_i| < 2$, I ∈ {1, 2, 3……., n}. Prove that for no z, |z| < $\frac{1}{3}$ and $\sum_{i=1}^{n} a_i z^i = 1$ can occur simultaneously.

[2]

Sol. $1 = \left|\sum a_i Z^i\right| \le \sum |a_i Z^i|$

$\Rightarrow \quad 1 \le |a_1 Z| + |a_2 Z^2| + |a_3 Z^3| + ... + |a_n Z^n|$

$< 2\left(|Z| + |Z|^2 + |Z|^3 + ... + |Z|^n\right)$

$\Rightarrow \quad 1 + |Z| + |Z|^2 + |Z|^3 + ….. + |Z|^n > 3/2$

Case I

$|Z| < 1$

$\Rightarrow \quad 1 + |Z| + |Z|^2 + ….. \infty > 3/2$

$\Rightarrow \quad \frac{1}{1-|Z|} > \frac{3}{2}$

$\Rightarrow \quad 2 > 3 - 3|Z|$

$\Rightarrow \quad |Z| > 1/3$

Case II

$|Z| \ge 1$, then
obviously, $|Z| < 1/3$ is not possible

Hence $|Z| < 1/3$ and $\sum_{i=1}^{n} a_i z^i = 1$ can not occur simultaneously for any a_i, $|a_i| < 2$.

7. If $f : [-2a, 2a] \rightarrow R$ be an odd function such that left hand derivative at x = a is zero and f(x) = f(2a – x), x ∈ (a, 2a), then find left hand derivative of f at x = –a.

[2]

Sol. $f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = 0$, x ∈ (0, 2a)

Now $f'(-a^-) = \lim_{h \to 0^-} \frac{f(-a+h) - f(-a)}{h}$

$= \lim_{h \to 0^-} \frac{-f(a-h) + f(a)}{h}$, f is an odd function.

$$= \lim_{h\to 0^-} \frac{-f(a+h)+f(a)}{h}$$, $f(x) = f(2a - x)$, $x \in (a, 2a)$

$$= -\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = 0$$

8. If $f(x)$ is an even function, then prove that

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$$

[2]

Sol.

$$I = \int_0^{\pi/2} f(\cos 2x)\cos x\,dx$$

$$I = \int_0^{\pi/2} f(\cos 2(\pi/2 - x))\cos(\pi/2 - x)dx$$

$$I = \int_0^{\pi/2} f(\cos 2x)\sin x\,dx$$, as f is even

$$2I = \int_0^{\pi/2} f(\cos 2x)(\cos x + \sin x)dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\cos 2x)\sin(x + \pi/4)dx$$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos(\pi/2 + 2t))\cos(t)dt$$, $x + \frac{\pi}{4} = \frac{\pi}{2} + t$

$$= \sqrt{2}\int_{-\pi 4}^{\pi/4} f(\sin 2t)\cos t\,dt$$

$$I = \sqrt{2}\int_0^{\pi/4} f(\sin 2t)\cos t\,dt$$

$$I = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$$.

9. A person has to go through three successive tests. Probability of his passing first exam is P. Probability of passing successive tests is P or P/2 according as he passed the last test or not. He is selected if he passes at least two tests. Find the probability of his selection.

[2]

Sol. Person is selected if either he passes all the tests or exactly two of the tests.
P (passing all the tests) = P.P.P = P^3
Probability of passing two tests
= P(first two tests) + P(first and third tests) + P(second and third tests)

$= P.P.(1-P) + P.(1-P).\frac{P}{2} + (1-P)\frac{P}{2}P$

$= P^2(1-P) + \frac{1}{2}P^2(1-P) + \frac{1}{2}P^2(1-P)$

$= 2P^2(1-P)$

Thus required probability = $P^3 + 2P^2(1-P) = 2P^2 - P^3$.

10. In a combat between *A*, *B* and *C*, *A* tries to hit *B* and *C*, and *B* and *C* try to hit *A*. Probability of A, B and C hitting the targets are 2/3, 1/2, and 1/3 respectively. If A is hit, find the probability that *B* hits *A* and *C* does not.

[2]

Sol. The required probability is given by

$$P(BC' \mid A) = \frac{P(A \mid BC').P(BC')}{P(A \mid BC')P(BC') + P(A \mid B'C).P(B'C) + P(A \mid BC).P(BC) + P(A \mid B'C').P(B'C')}$$

$$= \frac{1.\frac{1}{2}\times\frac{2}{3}}{1.\frac{1}{2}\times\frac{2}{3} + 1.\frac{1}{2}\times\frac{1}{3} + 1.\frac{1}{2}\times\frac{1}{3} + 0\times\frac{1}{2}\times\frac{2}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}+\frac{1}{6}} = \frac{1}{2}$$

11. Three normals with slopes m_1, m_2 and m_3 are drawn from a point P not on the axis of the parabola $y^2 = 4x$. If $m_1m_2 = \alpha$, results in the locus of P being a part of the parabola, find the value of α.

[4]

Sol. Any normal of slope m to the parabola $y^2 = 4x$ is

$y = mx - 2m - m^3$ (1)

If it passes through (h, k), then

$k = mh - 2m - m^3$

$\Rightarrow$ $m^3 + (2-h)m + k = 0$ (2)

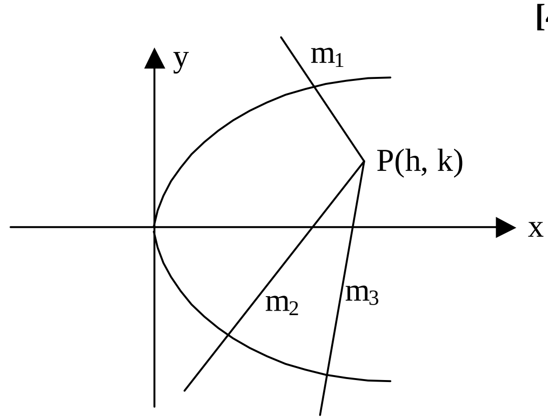
Thus $m_1m_2m_3 = -k$.

Now $m_1m_2 = \alpha \Rightarrow m_3 = \frac{-k}{\alpha}$

Now m_3 satisfies (2), so

$$-\frac{k^3}{\alpha^3} - (2-h)\frac{k}{\alpha} + k = 0$$

$\Rightarrow$ $k^3 + (2-h)k\alpha^2 - k\alpha^3 = 0$



Thus locus of P is

$y^3 + (2 - x)y\alpha^2 - y\alpha^3 = 0$

$\Rightarrow$ $y^2 + (2 - x)\alpha^2 - \alpha^3 = 0$, as $y \neq 0$ (P does not lie on the axis of the parabola)

$\Rightarrow$ $y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$

If it is a part of the parabola $y^2 = 4x$, then $\alpha^2 = 4$ and $-2\alpha^2 + \alpha^3 = 0$

$\Rightarrow$ $\alpha = 2$

12. Let $f : [0, 4] \to R$ be a differentiable function

(i) For some a, b $\in$ (0, 4), show that $f^2(4) - f^2(0) = 8f(a).f'(b)$

(ii) Show that $\int_0^4 f(t)dt = 2\left[\alpha f(\alpha^2) + \beta f(\beta^2)\right]$, for some $0 < \alpha; \beta < 2$.

[4]

Sol. (i) Using mean value theorem, there exists b $\in$ (0, 4) such that

$$f'(b) = \frac{f(4) - f(0)}{4} \qquad (1)$$

Now $\left(f(4)\right)^2 - \left(f(0)\right)^2 = \frac{\left(f(4) - f(0)\right)}{4}\left(f(4) + f(0)\right) \times 4$

From (1)

$$\left(f(4)\right)^2 - \left(f(0)\right)^2 = f'(b)\left(f(4) + f(0)\right) \times 4$$

Hence it is sufficient to prove that

$$\frac{f(0) + f(4)}{2} = f(a)$$

Range of function f must contain the interval $[f(0), f(4)]$ or $[f(4), f(0)]$ according as $f(0) \le f(4)$ or $f(0) \ge f(4)$

Now $\frac{f(0) + f(4)}{2}$ is the mean value of $f(0)$ and $f(4)$

$\Rightarrow$ $\left(\frac{f(0) + f(4)}{2}\right) \in$ range of the function

$\Rightarrow$ a $\in$ [0, 4] for which $f(a) = \frac{f(0) + f(4)}{2}$. Hence proved.

(ii) Let $\sqrt{t} = x \Rightarrow t = x^2 \Rightarrow dt = 2x\ dx$.

Thus $\int_0^4 f(t)dt = 2\int_0^2 xf(x^2)dx = 2(2 - 0)f(\varepsilon)$ for some $\varepsilon \in$ (0, 2), (using mean value theorem for definite integral of a differentiable function).

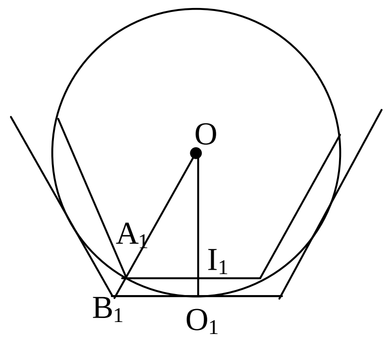
Thus $\int_0^4 f(t)dt = 2\left(f(\varepsilon) + f(\varepsilon)\right)$

$= 2\left(\alpha f(\alpha^2) + \beta f(\beta^2)\right)$, where $\alpha = \beta = \varepsilon$.

13. If I_n represents area of n–sided regular polygon inscribed in a unit circle and O_n the area of the n–sided regular polygon circumscribing it, prove that $I_n = \frac{O_n}{2}\left[1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right]$

[4]

Sol. $I_n = 2n \times$ area of ΔOA_1I_1



$\angle A_1OI_1 = \pi/n$

$= 2n \times \frac{1}{2} \times A_1I_1 \times OI_1$

$= n \times \sin\frac{\pi}{n} \times \cos\frac{\pi}{n}$

$= \frac{n}{2}\sin\frac{2\pi}{n}$.

$O_n = 2n \times$ area of $\Delta\, OB_1O_1$

$= 2n \times \frac{1}{2} \times B_1O_1 \times O_1O$

$= n \times \tan\frac{\pi}{n} \times 1 \quad = n\tan\frac{\pi}{n}$

Now R.H.S. $= \frac{O_n}{2}\left[1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right]$

$= \frac{O_n}{2}\left[1+\sqrt{1-\sin^2\frac{2\pi}{n}}\right] \quad = \frac{O_n}{2}\left[1+\cos\frac{2\pi}{n}\right]$

$= \frac{O_n}{2} \times 2\cos^2\frac{\pi}{n} \quad = O_n.\cos^2\frac{\pi}{n}$

$= n\tan\frac{\pi}{n}.\cos^2\frac{\pi}{n} \quad = \frac{n}{2}\sin\frac{2\pi}{n} \quad = I_n$.

Hence proved

14. Find the equation of the plane passing through (2, 1, 0); (4, 1, 1); (5, 0, 1). Find the point Q such that its distance from the plane is equal to the distance of point P(2, 1, 6) from the plane and the line joining P and Q is perpendicular to the plane.

[4]

Sol. Let equation of the plane be

$ax + by + cz + d = 0 \quad (1)$

(1) passes through the points (2, 1, 0); (4, 1, 1); (5, 0, 1)

a = –d/3; b = –d/3; c = $\frac{2}{3}d$

x + y – 2z – 3 = 0 (2)

which is the required equation of the plane

obviously Q is the image of P in the plane. It is easy to see that Q ≡ (6, 5, –2)

15. If $\hat{u}, \hat{v}, \hat{w}$ be three non–coplanar unit vectors with angles between $\hat{u}$ and $\hat{v}$ is α, between $\hat{v}$ and $\hat{w}$ is β and between $\hat{w}$ and $\hat{u}$ is γ. If $\vec{a}, \vec{b}, \vec{c}$ are the unit vectors along angle bisectors of α, β, γ respectively, then prove that $\left[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}\right] = \frac{1}{16}[\hat{u}\ \hat{v}\ \hat{w}]^2 \sec^2\left(\frac{\alpha}{2}\right)\sec^2\left(\frac{\beta}{2}\right)\sec^2\left(\frac{\gamma}{2}\right)$.

[4]

Sol.

$$\vec{a} = \frac{(\hat{u}+\hat{v})}{2|\cos\alpha/2|}$$

$$\vec{b} = \frac{(\hat{v}+\hat{w})}{2|\cos\beta/2|}$$

$$\vec{c} = \frac{(\hat{w}+\hat{u})}{2|\cos\gamma/2|}$$

∴ $\left[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}\right] = \left[\vec{a},\ \vec{b},\ \vec{c}\right]^2$ $= \left[\frac{((\hat{u}+\hat{v})\times(\hat{v}+\hat{w})).(\hat{w}+\hat{u})}{8|\cos\alpha/2.\cos\beta/2.\cos\gamma/2|}\right]^2$

$= \frac{[\hat{u}\ \hat{v}\ \hat{w}]^2}{16}\sec^2(\alpha/2).\sec^2(\beta/2).\sec^2(\gamma/2)$

16. If a, b and c are in arithmetic progression and a^2, b^2 and c^2 are in Harmonic progression, then prove that either a = b = c or a, b and –c/2 are in Geometric Progression.

[4]

Sol. Given that 2b = a + c (1)

a^2, b^2, c^2 are in H.P.

and $b^2 = \frac{2a^2c^2}{a^2+c^2}$ (2)

From (2) $b^2 = \frac{2a^2c^2}{4b^2-2ac}$, using (1)

⇒ $(ac - b^2)(ac + 2b^2) = 0$

⇒ $b^2 = ac$ or $2b^2 = -ac$.

Case I: $b^2 = ac$

⇒ $\left(\frac{a+c}{2}\right)^2 = ac$, using (1)

⇒ a = c

⇒ a = b = c, as a, b, c are in A.P.

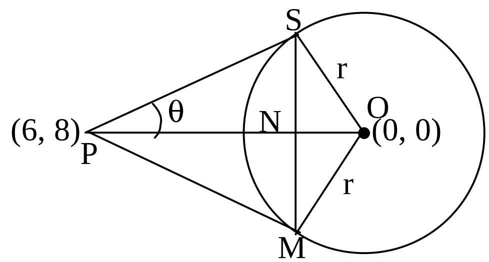
Case II: $2b^2 = -ac$

⇒ a, b, –c/2 are in G.P. (one of the possibilities)

17. Tangents are drawn from P(6, 8) to the circle $x^2 + y^2 = r^2$. Find the radius of the circle such that the area of the Δ formed by tangents and chord of contact is maximum.

[4]

Sol.



$\tan\theta = \dfrac{r}{PS}$

$= \dfrac{r}{\sqrt{100 - r^2}}$

$\sin\theta = \dfrac{r}{10}$

$\cos\theta = \dfrac{\sqrt{100 - r^2}}{10}$

Area of ΔPSM = $\dfrac{1}{2}$ SM × PN

$= \dfrac{1}{2}.2SN \times PN = SN \times PN$

= SPsinθ × SPcosθ

$= \left(\sqrt{100 - r^2}\right)^2 \times \sin\theta\cos\theta$

$= \left(100 - r^2\right)\dfrac{r}{10}\dfrac{\sqrt{100 - r^2}}{10}$

$= \dfrac{r\left(100 - r^2\right)^{3/2}}{100}$

$\dfrac{d\Delta}{dr} = \dfrac{1}{100} \cdot \dfrac{3}{2}\left(100 - r^2\right)^{1/2}(-2r)r + \dfrac{\left(100 - r^2\right)^{3/2}}{100} = 0$

$\left(100 - r^2\right)^{1/2}\left(-3r^2 + 100 - r^2\right) = 0$, $r \neq 10$ as P is outside the circle.

$4r^2 = 100 \Rightarrow r^2 = 25 \Rightarrow r = 5$

Thus for r = 5, Δ would be maximum.

18. $x^2 + (a - b)x + (1 - a - b) = 0$, a, b ∈ R. Find the condition on *a*, for which both roots of the equation are real and unequal.

[4]

Sol. For real and unequal roots, D > 0

$(a - b)^2 - 4(1 - a - b) > 0, \forall\, b \in R$

⇒ $b^2 - 2ab + 4b + a^2 + 4a - 4 > 0, \forall\, b \in R$

⇒ $b^2 - 2(a-2)b + a^2 + 4a - 4 > 0, \forall b \in R$
⇒ $4(a-2)^2 - 4(a^2 + 4a - 4) < 0$
⇒ $a^2 - 4a + 4 - a^2 - 4a + 4 < 0$
⇒ $8a > 8 \Rightarrow a > 1$.

19. Using $2(1 - \cos x) \le x, \forall x \in [0, \pi/4]$ or otherwise prove that $\sin(\tan x) \ge x, \forall x \in [0, \pi/4]$

[4]

Sol. Let $f(x) = \sin x - \tan^{-1} x$

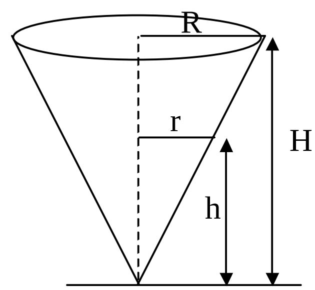
⇒ $f'(x) = \cos x - \frac{1}{1+x^2}$

Now in the first quadrant cosx is concave down and $\frac{1}{1+x^2}$ is concave up,

hence $f'(x) \ge 0$.
Thus f is an increasing function.
Hence $f(x) \ge f(0), \forall x \ge 0, x \le 1$
⇒ $\sin x \ge \tan^{-1} x$.
on replacing x by tanx, we get
$\sin(\tan x) \ge x$.
Hence proved.

20. An inverted cone of height H, and radius R is pointed at bottom. It is filled with a volatile liquid completely. If the rate of evaporation is directly proportional to the surface area of the liquid in contact with air (constant of proportionality k > 0). Find the time in which whole liquid evaporates.

[4]



Sol. $\frac{R}{H} = \frac{r}{h}$

$h = \frac{Hr}{R}$

$\frac{dv}{dt} = -k\pi r^2$

⇒ $\frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right) = -k\pi r^2$

⇒ $\frac{d}{dt}\left(\frac{r^3 H}{R}\right) = -3Kr^2$

⇒ $\frac{dr}{dt} = -k\frac{R}{H}$

$\int_R^0 dr = -\frac{kR}{H}\int_0^t dt$

$\Rightarrow \quad -R = \dfrac{-KR}{H}t \qquad \Rightarrow \quad t = \dfrac{H}{k}$